Two generalizations of the Busche-Ramanujan identities

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Abstract

We derive two new generalizations of the Busche-Ramanujan identities involving the multiple Dirichlet convolution of arithmetic functions of several variables. The proofs use formal multiple Dirichlet series and properties of symmetric polynomials of several variables.

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1 Introduction

Throughout the paper we use the notation: $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $\mathbf{1}(n) = 1$, $\mathrm{id}(n) = n$, $\mathrm{id}_k(n) = n^k$ ($k \in \mathbb{C}$, $n \in \mathbb{N}$), μ is the Möbius function, $*$ denotes the Dirichlet convolution of arithmetic functions, ζ is the Riemann zeta function.

Let g and h be two completely multiplicative arithmetic functions and let $f = g * h$. The Busche-Ramanujan identities state that for every $m, n \in \mathbb{N}$,

$$f(mn) = \sum_{a \mid \gcd(m,n)} f\left(\frac{m}{a}\right) f\left(\frac{n}{a}\right) \mu(a) g(a) h(a) \tag{1}$$

and

$$f(m)f(n) = \sum_{a \mid \gcd(m,n)} f\left(\frac{mn}{a^2}\right) g(a) h(a). \tag{2}$$

For example, these identities hold true for the following special functions f: (i) the function $\sigma_k = \mathbf{1} * \mathrm{id}_k$, in particular, the divisor function $d = \mathbf{1} * \mathbf{1}$ and the sum-of-divisors function $\sigma = \mathbf{1} * \mathrm{id}$; (ii) the alternating sum-of-divisors function $\beta = \lambda * \mathrm{id}$, where λ is the Liouville function, cf. [19]; (iii) the function $R_1 = \mathbf{1} * \chi$, where χ is the nonprincipal character (mod 4), $R(n) = 4R_1(n)$ representing the number of ordered pairs $(x, y) \in \mathbb{Z}^2$ such that $n = x^2 + y^2$; (iv) Ramanujan's function τ, defined by the expansion

$$\sum_{n=1}^{\infty} \tau(n) x^n = x \prod_{n=1}^{\infty} (1 - x^n)^{24}, \quad |x| < 1,$$

where $\tau = g_1 * g_2$ with certain completely multiplicative functions g_1, g_2 verifying $g_1(n)g_2(n) = n^{11}$ for every $n \in \mathbb{N}$, cf., e.g., [2, 9].

For history and discussion, as well as for generalizations and analogues of the Busche-Ramanujan identities we refer to [3, 5, 6, 10, 11, 12, 13, 14, 17, 20]. The functions which

are the convolution of two completely multiplicative functions are called in the literature specially multiplicative or quadratic functions. See the papers [7, 15, 16] for various other properties of such functions.

In the present paper we derive two new generalizations of the Busche-Ramanujan identities. Namely, we consider the values of a specially multiplicative function for products of several arbitrary integers (Theorem 1), and then deduce formulae for the convolution of several arbitrary completely multiplicative functions (Theorem 2). Our general formulae involve multiplicative functions of several variables and their Dirichlet convolution. Such a treatment explains also the equivalence of the identities of type (1) and (2). The proofs use simple arguments concerning formal multiple Dirichlet series of arithmetic functions of several variables and properties of symmetric polynomials of several variables.

2 Preliminaries

In the paper we use the following notions and properties. Let $\mathcal{F}_r$ be the set of arithmetic functions of r ($r \in \mathbb{N}$) variables, i.e., of functions $f : \mathbb{N}^r \to \mathbb{C}$. If $f, g \in \mathcal{F}_r$, then their Dirichlet convolution is defined by

$$(f * g)(n_1, \ldots, n_r) = \sum_{a_1 \mid n_1, \ldots, a_r \mid n_r} f(a_1, \ldots, a_r) g(n_1/a_1, \ldots, n_r/a_r).$$

The set $\mathcal{F}_r$ forms a commutative ring with identity δ, where $\delta(1, \ldots, 1) = 1$ and $\delta(n_1, \ldots, n_r) = 0$ for $n_1 \cdots n_r > 1$. A function $f \in \mathcal{F}_r$ has an inverse under the convolution, denoted by f^{-1*}, if and only if $f(1, \ldots, 1) \neq 0$.

A function $f \in \mathcal{F}_r$ is said to be multiplicative if it is not identically zero and

$$f(m_1 n_1, \ldots, m_r n_r) = f(m_1, \ldots, m_r) f(n_1, \ldots, n_r)$$

holds for any $m_1, \ldots, m_r, n_1, \ldots, n_r \in \mathbb{N}$ such that $\gcd(m_1 \cdots m_r, n_1 \cdots n_r) = 1$. If f is multiplicative, then it is determined by the values $f(p^{\nu_1}, \ldots, p^{\nu_r})$, where p is prime and $\nu_1, \ldots, \nu_r \in \mathbb{N}_0$. More exactly, $f(1, \ldots, 1) = 1$ and for any $n_1, \ldots, n_r \in \mathbb{N}$,

$$f(n_1, \ldots, n_r) = \prod_p f(p^{\nu_p(n_1)}, \ldots, p^{\nu_p(n_r)}),$$

where $n_i = \prod_p p^{\nu_p(n_i)}$ is the prime power factorization of n_i ($1 \le i \le r$), the products being over the primes p and all but a finite number of the exponents $\nu_p(n_i)$ are zero.

The convolution of multiplicative functions is multiplicative. The inverse of a multiplicative function is multiplicative. If $f \in \mathcal{F}_1$ is multiplicative, then the function $(n_1, \ldots, n_r) \mapsto f(n_1 \cdots n_r)$ is multiplicative.

The (formal) multiple Dirichlet series of a function $f \in \mathcal{F}_r$ is given by

$$D(f; s_1, \ldots, s_r) = \sum_{n_1, \ldots, n_r = 1}^{\infty} \frac{f(n_1, \ldots, n_r)}{n_1^{s_1} \cdots n_r^{s_r}}.$$

We have

$$D(f * g; s_1, \ldots, s_r) = D(f; s_1, \ldots, s_r) D(g; s_1, \ldots, s_r),$$

and

$$D(f^{-1*}; s_1, \ldots, s_r) = D(f; s_1, \ldots, s_r)^{-1},$$

formally or in the case of absolute convergence.

If $f \in \mathcal{F}_r$ is multiplicative, then its Dirichlet series can be expanded into a (formal) Euler product, that is,

$$D(f; s_1, \ldots, s_r) = \prod_p \sum_{\nu_1, \ldots, \nu_r = 0}^{\infty} \frac{f(p^{\nu_1}, \cdots, p^{\nu_r})}{p^{\nu_1 s_1 + \ldots + \nu_r s_r}},$$

the product being over the primes p.

If $r = 1$, i.e., in the case of functions of a single variable we reobtain the familiar notions and properties, cf. [1, 12, 17]. See [17, Ch. VII], [18], [20] for the case of several variables.

We also need that in the one variable case, if g and h are completely multiplicative and $f = g * h$, then for every prime power p^ν ($\nu \in \mathbb{N}$),

$$f(p^\nu) = \begin{cases} \frac{g(p)^{\nu+1} - h(p)^{\nu+1}}{g(p) - h(p)}, & g(p) \neq h(p), \\ (\nu+1) g(p)^\nu, & g(p) = h(p). \end{cases} \tag{3}$$

Furthermore, if g is completely multiplicative, then

$$D(g; s) = \prod_p \left(1 - \frac{g(p)}{p^s}\right)^{-1}. \tag{4}$$

3 Results

We prove the following results.

Theorem 1. *Let g and h be two completely multiplicative functions and let $f = g * h$. Let ψ_f be the multiplicative function of r ($r \geq 2$) variables defined as follows. For every prime p and every $\nu_1, \ldots, \nu_r \in \mathbb{N}_0$ set*

$$\psi_f(p^{\nu_1}, \ldots, p^{\nu_r}) = \begin{cases} 1, & \nu_1 = \ldots = \nu_r = 0, \\ (-1)^{j-1} g(p) h(p) f(p^{j-2}), & \nu_1, \ldots, \nu_r \in \{0, 1\}, j := \nu_1 + \ldots + \nu_r \geq 2, \\ 0, & \textit{otherwise}. \end{cases}$$

Furthermore, let ψ_f^{-1*} be the inverse under the r variables convolution of the function ψ_f. Then for every $n_1, \ldots, n_r \in \mathbb{N}$ ($r \geq 2$),

$$f(n_1 \cdots n_r) = \sum_{a_1 \mid n_1, \ldots, a_r \mid n_r} f\left(\frac{n_1}{a_1}\right) \cdots f\left(\frac{n_r}{a_r}\right) \psi_f(a_1, \ldots, a_r), \tag{5}$$

and

$$f(n_1) \cdots f(n_r) = \sum_{a_1 \mid n_1, \ldots, a_r \mid n_r} f\left(\frac{n_1 \cdots n_r}{a_1 \cdots a_r}\right) \psi_f^{-1*}(a_1, \ldots, a_r). \tag{6}$$

Theorem 2. *Let $f_1, \ldots, f_k$ be completely multiplicative functions ($k \geq 2$) and let $F = f_1 * \cdots * f_k$. Let ϑ_F be the multiplicative function of two variables defined as follows. For every prime p and every $\nu_1, \nu_2 \in \mathbb{N}_0$ set*

$$\vartheta_F(p^{\nu_1}, p^{\nu_2}) = \begin{cases} 1, & \nu_1 = \nu_2 = 0, \\ (-1)^{\nu_1 + \nu_2 - 1} e_{\nu_1 + \nu_2}(f_1(p), \ldots, f_k(p)), & \nu_1, \nu_2 \geq 1, \nu_1 + \nu_2 \leq k, \\ 0, & \textit{otherwise}, \end{cases}$$

where $e_d(x_1,\ldots,x_k)$ *represents the elementary symmetric polynomial in* $x_1,\ldots,x_k$ *of degree* d. *Furthermore, let* ϑ_F^{-1*} *denote the inverse under the two variables convolution of the function* ϑ_F. *Then for every* $n_1,n_2\in\mathbb{N}$,

$$F(n_1n_2)=\sum_{a_1\mid n_1,\,a_2\mid n_2} F\left(\frac{n_1}{a_1}\right)F\left(\frac{n_2}{a_2}\right)\vartheta_F(a_1,a_2), \tag{7}$$

and

$$F(n_1)F(n_2)=\sum_{a_1\mid n_1,\,a_2\mid n_2} F\left(\frac{n_1n_2}{a_1a_2}\right)\vartheta_F^{-1*}(a_1,a_2). \tag{8}$$

Theorems 1 and 2 reduce in the cases $r=2$, respectively $k=2$ to the Busche-Ramanujan identities (1) and (2).

For Ramanujan's tau function and for the Piltz divisor function d_k we obtain from our results the next identities. We recall that $d_k(n)$ is the number of ordered k-tuples $(x_1,\ldots,x_k)$ of positive integers such that $x_1\cdots x_k=n$.

Corollary 1. *For every* $n_1,\ldots,n_r\in\mathbb{N}$ $(r\ge 2)$,

$$\tau(n_1\cdots n_r)=\sum_{a_1\mid n_1,\,\ldots,\,a_r\mid n_r}\tau\left(\frac{n_1}{a_1}\right)\cdots\tau\left(\frac{n_r}{a_r}\right)\psi_\tau(a_1,\ldots,a_r),$$

where the multiplicative function ψ_τ *is defined for every prime* p *and every* $\nu_1,\ldots,\nu_r\in\mathbb{N}_0$ *by*

$$\psi_\tau(p^{\nu_1},\ldots,p^{\nu_r})=\begin{cases}1, & \nu_1=\ldots=\nu_r=0,\\ (-1)^{j-1}p^{11}\tau(p^{j-2}), & \nu_1,\ldots,\nu_r\in\{0,1\},\, j:=\nu_1+\ldots+\nu_r\ge 2,\\ 0, & \textit{otherwise.}\end{cases}$$

Corollary 2. *Let* $k\ge 2$. *For every* $n_1,n_2\in\mathbb{N}$,

$$d_k(n_1n_2)=\sum_{a_1\mid n_1,\,a_2\mid n_2} d_k\left(\frac{n_1}{a_1}\right)d_k\left(\frac{n_2}{a_2}\right)\vartheta_k(a_1,a_2),$$

where the multiplicative function ϑ_k *is defined for every prime* p *and every* $\nu_1,\nu_2\in\mathbb{N}_0$ *by*

$$\vartheta_k(p^{\nu_1},p^{\nu_2})=\begin{cases}1, & \nu_1=\nu_2=0,\\ (-1)^{\nu_1+\nu_2-1}\binom{k}{\nu_1+\nu_2}, & \nu_1,\nu_2\ge 1,\nu_1+\nu_2\le k,\\ 0, & \textit{otherwise.}\end{cases}$$

4 Proof of Theorem 1

Formula (5) is a direct consequence of the following identity concerning multiple Dirichlet series: If g and h are completely multiplicative functions and $f=g*h$, then

$$\sum_{n_1,\ldots,n_r=1}^{\infty}\frac{f(n_1\cdots n_r)}{n_1^{s_1}\cdots n_r^{s_r}}=\left(\sum_{n_1=1}^{\infty}\frac{f(n_1)}{n_1^{s_1}}\right)\cdots\left(\sum_{n_r=1}^{\infty}\frac{f(n_r)}{n_r^{s_r}}\right)$$

$$\times\prod_p\left(1+g(p)h(p)\sum_{j=2}^{r}(-1)^{j-1}f(p^{j-2})\sum_{1\le i_1<\ldots<i_j\le r}\frac{1}{p^{s_{i_1}+\ldots+s_{i_j}}}\right). \tag{9}$$

To obtain this result use (3) and (4). Since the function $(n_1,\ldots,n_r)\mapsto f(n_1\cdots n_r)$ is multiplicative, we deduce

$$D:=\sum_{n_1,\ldots,n_r=1}^{\infty}\frac{f(n_1\cdots n_r)}{n_1^{s_1}\cdots n_r^{s_r}}=\prod_p\sum_{\nu_1,\ldots,\nu_r=0}^{\infty}\frac{f(p^{\nu_1+\ldots+\nu_r})}{p^{\nu_1 s_1+\ldots+\nu_r s_r}}$$

$$=\prod_p\sum_{\nu_1,\ldots,\nu_r=0}^{\infty}\frac{g(p)^{\nu_1+\ldots+\nu_r+1}-h(p)^{\nu_1+\ldots+\nu_r+1}}{(g(p)-h(p))p^{\nu_1 s_1+\ldots+\nu_r s_r}}$$

using the convention that if $g(p)=h(p)$ for some primes p, then we consider the limit values (for $h(p)\to g(p)$) of the corresponding fractions, i.e., the second formula of (3).

Therefore,

$$D=\prod_p\frac{1}{g(p)-h(p)}\left(g(p)\sum_{\nu_1,\ldots,\nu_r=0}^{\infty}\left(\frac{g(p)}{p^{s_1}}\right)^{\nu_1}\cdots\left(\frac{g(p)}{p^{s_r}}\right)^{\nu_r}-h(p)\sum_{\nu_1,\ldots,\nu_r=0}^{\infty}\left(\frac{h(p)}{p^{s_1}}\right)^{\nu_1}\cdots\left(\frac{h(p)}{p^{s_r}}\right)^{\nu_r}\right)$$

$$=\prod_p\frac{1}{g(p)-h(p)}\left(g(p)\left(1-\frac{g(p)}{p^{s_1}}\right)^{-1}\cdots\left(1-\frac{g(p)}{p^{s_r}}\right)^{-1}-h(p)\left(1-\frac{h(p)}{p^{s_1}}\right)^{-1}\cdots\left(1-\frac{h(p)}{p^{s_r}}\right)^{-1}\right)$$

$$=\prod_p\frac{1}{g(p)-h(p)}\left(1-\frac{g(p)}{p^{s_1}}\right)^{-1}\left(1-\frac{h(p)}{p^{s_1}}\right)^{-1}\cdots\left(1-\frac{g(p)}{p^{s_r}}\right)^{-1}\left(1-\frac{h(p)}{p^{s_r}}\right)^{-1}$$

$$\times\left(g(p)\left(1-\frac{h(p)}{p^{s_1}}\right)\cdots\left(1-\frac{h(p)}{p^{s_r}}\right)-h(p)\left(1-\frac{g(p)}{p^{s_1}}\right)\cdots\left(1-\frac{g(p)}{p^{s_r}}\right)\right)$$

$$=D(g,s_1)D(h,s_1)\cdots D(g,s_r)D(h,s_r)$$

$$\times\prod_p\left(1-g(p)h(p)\sum_{1\le i<j\le r}\frac{1}{p^{s_i+s_j}}+g(p)h(p)\left(g(p)+h(p)\right)\sum_{1\le i<j<k\le r}\frac{1}{p^{s_i+s_j+s_k}}-\ldots\right.$$

$$\left.+(-1)^{r-1}g(p)h(p)\left(g(p)^{r-2}+g(p)^{r-3}h(p)+\ldots+g(p)h(p)^{r-3}+h(p)^{r-2}\right)\frac{1}{p^{s_1+\ldots+s_r}}\right),$$

simplifying by $g(p)-h(p)$. The computations are valid also in the case when $g(p)=h(p)$ for some primes p. This gives (9). Formula (6) is obtained by expressing the function $(n_1,\ldots,n_r)\mapsto f(n_1)\cdots f(n_r)$ from the convolutional identity (5). The proof is finished.

Note that for the divisor function d formula (9) gives

$$\sum_{n_1,\ldots,n_r=1}^{\infty}\frac{d(n_1\cdots n_r)}{n_1^{s_1}\cdots n_r^{s_r}}=\zeta^2(s_1)\cdots\zeta^2(s_r)\prod_p\left(1+\sum_{j=2}^{r}(-1)^{j-1}(j-1)\sum_{1\le i_1<\ldots<i_j\le r}\frac{1}{p^{s_{i_1}+\ldots+s_{i_j}}}\right),$$

which reduces in the case $r=2$ to

$$\sum_{n_1,n_2=1}^{\infty}\frac{d(n_1n_2)}{n_1^{s_1}n_2^{s_2}}=\frac{\zeta^2(s_1)\zeta^2(s_2)}{\zeta(s_1+s_2)}.$$

These identities corresponding to the cases $r=2$ and $r=3$ were pointed out in [8, Sect. 6]. We also remark that for the σ function the common analytic version of (1) and (2) is the formula

$$\sum_{n_1,n_2=1}^{\infty}\frac{\sigma(n_1n_2)}{n_1^{s_1}n_2^{s_2}}=\frac{\zeta(s_1)\zeta(s_1-1)\zeta(s_2)\zeta(s_2-1)}{\zeta(s_1+s_2-1)}.$$

5 Proof of Theorem 2

Let $e_d(x_1,\ldots,x_k)=\sum_{1\le i_1<\ldots<i_d\le k} x_{i_1}\cdots x_{i_d}$ be the elementary symmetric polynomial in x_1, $\ldots,x_k$ of degree d ($d\ge 0$). By convention, $e_0(x_1,\ldots,x_k)=1$ and $e_d(x_1,\ldots,x_k)=0$ ($d\ge k+1$). Furthermore, let $h_d(x_1,\ldots,x_k)=\sum_{1\le i_1\le\ldots\le i_d\le k} x_{i_1}\cdots x_{i_d}$ stand for the complete homogeneous symmetric polynomial in $x_1,\ldots,x_k$ of degree d ($d\ge 0$). By convention, $h_0(x_1,\ldots,x_k)=1$. We use the representation

$$h_d(x_1,\ldots,x_k)=\sum_{i=1}^{k} x_i^{d+k-1}\prod_{\substack{j=1\\ j\ne i}}^{k}(x_i-x_j)^{-1} \quad (d\ge 0), \tag{10}$$

cf. [4]. We also need the well known polynomial identity

$$\sum_{d=0}^{n}(-1)^d e_d(x_1,\ldots,x_k)h_{n-d}(x_1,\ldots,x_k)=\begin{cases}1, & n=0,\\ 0, & n\ge 1.\end{cases} \tag{11}$$

If $f_1,\ldots,f_k$ are completely multiplicative functions and $F=f_1*\cdots*f_k$, then for every prime power p^ν ($\nu\in\mathbb{N}$),

$$F(p^\nu)=h_\nu(x_1,\ldots,x_k),$$

where $x_i=f_i(p)$ ($1\le i\le k$). Therefore,

$$\sum_{n_1,n_2=1}^{\infty}\frac{F(n_1n_2)}{n_1^{s_1}n_2^{s_2}}=\prod_p\sum_{\nu_1,\nu_2=0}^{\infty}\frac{F(p^{\nu_1+\nu_2})}{p^{\nu_1s_1+\nu_2s_2}}$$

$$=\prod_p\left(\sum_{\nu_1,\nu_2=0}^{\infty}\frac{1}{p^{\nu_1s_1+\nu_2s_2}}\sum_{i=1}^{k}x_i^{\nu_1+\nu_2+k-1}\prod_{\substack{j=1\\ j\ne i}}^{k}(x_i-x_j)^{-1}\right)$$

$$=\prod_p\left(\sum_{i=1}^{k}x_i^{k-1}\prod_{\substack{j=1\\ j\ne i}}^{k}(x_i-x_j)^{-1}\sum_{\nu_1,\nu_2=0}^{\infty}\left(\frac{x_i}{p^{s_1}}\right)^{\nu_1}\left(\frac{x_i}{p^{s_2}}\right)^{\nu_2}\right)$$

$$=\prod_p\left(\sum_{i=1}^{k}x_i^{k-1}\left(1-\frac{x_i}{p^{s_1}}\right)^{-1}\left(1-\frac{x_i}{p^{s_2}}\right)^{-1}\prod_{\substack{j=1\\ j\ne i}}^{k}(x_i-x_j)^{-1}\right)$$

$$=\prod_p\left(\prod_{\ell=1}^{k}\left(1-\frac{x_\ell}{p^{s_1}}\right)^{-1}\left(1-\frac{x_\ell}{p^{s_2}}\right)^{-1}\sum_{i=1}^{k}x_i^{k-1}\prod_{\substack{j=1\\ j\ne i}}^{k}(x_i-x_j)^{-1}\left(1-\frac{x_j}{p^{s_1}}\right)\left(1-\frac{x_j}{p^{s_2}}\right)\right)$$

$$=\left(\sum_{n_1=1}^{\infty}\frac{F(n_1)}{n_1^{s_1}}\right)\left(\sum_{n_2=1}^{\infty}\frac{F(n_r)}{n_2^{s_2}}\right)\prod_p Q_k(p^{-s_1},p^{-s_2}),$$

where $Q_k(u,v)$ is the polynomial in u and v, given by

$$Q_k(u,v)=\sum_{i=1}^{k}x_i^{k-1}\prod_{\substack{j=1\\ j\ne i}}^{k}(x_i-x_j)^{-1}\left(1-x_ju\right)\left(1-x_jv\right)=\sum_{m,n=0}^{k-1}c_{m,n}u^mv^n.$$

Here the coefficients $c_{m,n}$ $(1 \le m, n \le k-1)$ are given by

$$c_{m,n} = (-1)^{m+n} \sum_{i=1}^{k} x_i^{k-1} \prod_{\substack{j=1 \\ j \neq i}}^{k} (x_i - x_j)^{-1} e_m^{(i)}(x_1, \ldots, x_k) e_n^{(i)}(x_1, \ldots, x_k) \tag{12}$$

where $e_m^{(i)}(x_1, \ldots, x_k) = e_m(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)$ $(1 \le i \le k)$. We will show that

$$c_{m,n} = \begin{cases} 1, & m = n = 0, \\ (-1)^{m+n-1} e_{m+n}(x_1, \ldots, x_k), & m, n \ge 1, m+n \le k \\ 0, & \text{otherwise.} \end{cases}$$

To this end, note that $e_m^{(i)}(x_1, \ldots, x_k) = e_m(x_1, \ldots, x_k) - x_i e_{m-1}^{(i)}(x_1, \ldots, x_k)$ $(1 \le m \le k)$, which leads to the identity

$$e_m^{(i)}(x_1, \ldots, x_k) = \sum_{\ell=0}^{m} (-1)^\ell x_i^\ell e_{m-\ell}(x_1, \ldots, x_k).$$

Therefore, from (12) we deduce

$$c_{m,n} = (-1)^{m+n} \sum_{i=1}^{k} x_i^{k-1} \prod_{\substack{j=1 \\ j \neq i}}^{k} (x_i - x_j)^{-1} \sum_{\ell=0}^{m} (-1)^\ell x_i^\ell e_{m-\ell}(x_1, \ldots, x_k) \sum_{s=0}^{n} (-1)^s x_i^s e_{n-s}(x_1, \ldots, x_k)$$

$$= (-1)^{m+n} \sum_{\ell=0}^{m} \sum_{s=0}^{n} (-1)^{\ell+s} e_{m-\ell}(x_1, \ldots, x_k) e_{n-s}(x_1, \ldots, x_k) \sum_{i=1}^{k} x_i^{\ell+s+k-1} \prod_{\substack{j=1 \\ j \neq i}}^{k} (x_i - x_j)^{-1}$$

$$= (-1)^{m+n} \sum_{\ell=0}^{m} (-1)^\ell e_{m-\ell}(x_1, \ldots, x_k) \sum_{s=0}^{n} (-1)^s e_{n-s}(x_1, \ldots, x_k) h_{\ell+s}(x_1, \ldots, x_k), \tag{13}$$

using again (10). For $n = 0$ this gives, by (11),

$$c_{m,0} = (-1)^m \sum_{\ell=0}^{m} (-1)^\ell e_{m-\ell}(x_1, \ldots, x_k) h_\ell(x_1, \ldots, x_k) = \begin{cases} 1, & m = 0, \\ 0, & m \ge 1. \end{cases}$$

We deduce in the same way that $c_{0,n} = 0$ for $n \ge 1$. Now let $m, n \ge 1$. Then the inner sum in (13) is, by denoting $j = \ell + s$,

$$\sum_{j=\ell}^{n+\ell} (-1)^{j-\ell} e_{n+\ell-j}(x_1, \ldots, x_k) h_j(x_1, \ldots, x_k)$$

$$= (-1)^\ell \left(\sum_{j=0}^{n+\ell} (-1)^j e_{n+\ell-j}(x_1, \ldots, x_k) h_j(x_1, \ldots, x_k) - \sum_{j=0}^{\ell-1} (-1)^j e_{n+\ell-j}(x_1, \ldots, x_k) h_j(x_1, \ldots, x_k) \right)$$

$$= (-1)^{\ell-1} \sum_{j=0}^{\ell-1} (-1)^j e_{n+\ell-j}(x_1, \ldots, x_k) h_j(x_1, \ldots, x_k), \tag{14}$$

since the first sum is zero, according to (11), where $n+\ell \geq 1+\ell \geq 1$ for every $\ell \geq 0$. For $\ell = 0$ (14) is zero (empty sum). We obtain

$$c_{m,n} = (-1)^{m+n-1} \sum_{\ell=1}^{m} e_{m-\ell}(x_1,\ldots,x_k) \sum_{j=0}^{\ell-1} (-1)^j e_{n+\ell-j}(x_1,\ldots,x_k) h_j(x_1,\ldots,x_k)$$

and regrouping the terms according to the values $t = \ell - j$,

$$c_{m,n} = (-1)^{m+n-1} \sum_{t=1}^{m} e_{n+t}(x_1,\ldots,x_k) \sum_{j=0}^{m-t} (-1)^j e_{m-t-j}(x_1,\ldots,x_k) h_j(x_1,\ldots,x_k),$$

where the inner sum is 0 for $t < m$ and it is 1 for $t = m$. Therefore,

$$c_{m,n} = (-1)^{m+n-1} e_{m+n}(x_1,\ldots,x_k),$$

which is zero for $m+n > k$. This finishes the proof of (7). Now (8) is obtained by expressing the function $(n_1,n_2) \mapsto F(n_1)F(n_2)$.

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